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Weak limit of iterates of some random-valued functions and its application

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Dedicated to Professor János Aczél on his 95th birthday.

Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$, a complete and separable metric space X with the σ-algebra $\mathcal{B}$ of all its Borel subsets, a $\mathcal{B} \otimes \mathcal{A}$-measurable and contractive in mean $f : X \times \Omega \to X$, and a Lipschitz F mapping X into a separable Banach space Y we characterize the solvability of the equation

$$\varphi(x) = \int_\Omega \varphi\left(f(x,\omega)\right) P(d\omega) + F(x)$$

in the class of Lipschitz functions $\varphi : X \to Y$ with the aid of the weak limit π^f of the sequence of iterates $(f^n(x,\cdot))_{n\in\mathbb{N}}$ of f, defined on $X \times \Omega^{\mathbb{N}}$ by $f^0(x,\omega) = x$ and $f^n(x,\omega) = f\left(f^{n-1}(x,\omega), \omega_n\right)$ for $n \in \mathbb{N}$, and propose a characterization of π^f for some special rv-functions in Hilbert spaces.

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1. Introduction

Fix a probability space $(\Omega, \mathcal{A}, P)$ and a complete and separable metric space (X, ρ).

Let $\mathcal{B}$ denote the σ-algebra of all Borel subsets of X. We say that $f : X \times \Omega \to X$ is a *random-valued* function (shortly: an *rv-function*) if it is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of such an rv-function are given by

$$f^0(x,\omega_1,\omega_2,\dots) = x, \quad f^n(x,\omega_1,\omega_2,\dots) = f\left(f^{n-1}(x,\omega_1,\omega_2,\dots),\omega_n\right)$$

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for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \ldots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n : X \times \Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, for $n \in \mathbb{N}$ the nth iterate f^n is $\mathcal{B} \otimes \mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the σ-algebra of all sets of the form

$$\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$. (See [8, Sec. 1.4], [6].)

Let $f : X \times \Omega \to X$ be an rv-function.

A result on the a.s. convergence of $(f^n(x, \cdot))_{n \in \mathbb{N}}$ for X being the unit interval may be found in [8, Sec. 1.4B]. The paper [6] by Rafał Kapica brings theorems on the convergence a.s. and in L^1 of those sequences of iterates in the case where X is a closed subset of a Banach lattice. A simple criterion for the convergence in law of $(f^n(x, \cdot))_{n \in \mathbb{N}}$ to a random variable independent of $x \in X$ was proved in [1] and applied to the equation

$$\varphi(x) = \int_\Omega \varphi\left(f(x, \omega)\right) P(d\omega) + F(x) \tag{1}$$

with φ as the unknown function. This criterion reads as follows.

(H) There exists a $\lambda \in (0, 1)$ such that

$$\int_\Omega \varrho\left(f(x, \omega), f(z, \omega)\right) P(d\omega) \leq \lambda \varrho(x, z) \quad \text{for } x, z \in X \tag{2}$$

and

$$\int_\Omega \varrho\left(f(x, \omega), x\right) P(d\omega) < \infty \quad \text{for } x \in X. \tag{3}$$

Thus, denoting by $\pi_n^f(x, \cdot)$ the distribution of $f^n(x, \cdot)$, i.e.,

$$\pi_n^f(x, B) = P^\infty\left(f^n(x, \cdot) \in B\right) \quad \text{for } n \in \mathbb{N} \cup \{0\},\ x \in X \text{ and } B \in \mathcal{B},$$

hypothesis (H) guarantees the existence of a probability Borel measure π^f on X such that

$$\lim_{n \to \infty} \int_X u(z) \pi_n^f(x, dz) = \int_X u(z) \pi^f(dz)$$

holds for $x \in X$ and for any continuous and bounded $u : X \to \mathbb{R}$; more exactly, cf. also [2, Theorem 3.1],

$$\int_X \varrho(x, z) \pi^f(dz) < \infty \quad \text{for } x \in X \tag{4}$$

and

$$\left| \int_X u(z) \pi_n^f(x, dz) - \int_X u(z) \pi^f(dz) \right| \leq \frac{\lambda^n}{1 - \lambda} \int_X \varrho\left(f(x, \omega), x\right) P(d\omega) \tag{5}$$

for $x \in X$, $n \in \mathbb{N}$ and a non-expansive u mapping X into $[-1, 1]$.

Rafał Kapica strengthened this estimation showing, see [7, Corollary 5.6 and Lemma 3.1], that (5) holds for $x \in X$, $n \in \mathbb{N}$ and a non-expansive $u : X \to \mathbb{R}$. Since it is explicitly stated there only for a non-expansive and bounded

$u : X \to \mathbb{R}$ we prove it for a non-expansive u mapping X into a separable Banach space making use of (5) for non-expansive and bounded $u : X \to \mathbb{R}$ only. Having done that we characterize the solvability of (1) in the class of Lipschitz functions with the aid of this limit distribution π^f. Moreover, we propose a characterization of π^f for some special rv-functions in Hilbert spaces.

2. Solvability of the equation

Following [3] given an rv-function $f : X \times \Omega \to X$ such that (H) holds and a Lipschitz F mapping X into a separable Banach space Y define

$$F_0(x) = F(x), \quad F_n(x) = \int_\Omega F_{n-1}\left(f(x,\omega)\right) P(d\omega)$$

for $x \in X$ and $n \in \mathbb{N}$, and note that according to [3, Theorem 2.1] there exists a $y_0 \in Y$ such that for every $x \in X$ the sequence $(F_n(x))_{n\in\mathbb{N}}$ converges to y_0, and any Lipschitz solution $\varphi\colon X \to Y$ of (1) has the form

$$\varphi(x) = c + \sum_{n=0}^{\infty} F_n(x) \quad \text{for } x \in X,$$

where c is a constant from Y. With the aid of the limit distribution π^f we characterize this pointwise limit of $(F_n)_{n\in\mathbb{N}}$ and, making use of [3, Theorem 2.1(ii)], the solvability of (1) as follows (cf. [1, Corollary 4.1]).

Theorem 2.1. *Assume* (H). *If F is a Lipschitz mapping of X into a separable Banach space Y, then*

$$\lim_{n\to\infty} F_n(x) = \int_X F(z)\pi^f(dz) \quad \textit{for } x \in X \tag{6}$$

and Eq. (1) *has a Lipschitz solution $\varphi : X \to Y$ if and only if*

$$\int_X F(z)\pi^f(dz) = 0. \tag{7}$$

As announced above, we start with the following lemma.

Lemma 2.2. *If $f : X \times \Omega \to X$ is an rv-function such that* (2) *holds with a $\lambda \in (0, 1)$ and* (3) *is satisfied, then*

$$\left\| \int_X u(z)\pi_n^f(x, dz) - \int_X u(z)\pi^f(dz) \right\| \le \frac{\lambda^n}{1-\lambda} \int_X \varrho\left(f(x,\omega), x\right) P(d\omega) \tag{8}$$

for $x \in X$, $n \in \mathbb{N}$ and for any non-expansive u mapping X into a separable Banach space.

Proof. First of all let us observe that for every $x \in X$, $n \in \mathbb{N}$ and $(\omega_1, \omega_2, \ldots) \in \Omega^\infty$ we have

$$\begin{aligned}\rho\left(f^n(x,\omega_1,\omega_2,\ldots),x\right) &= \rho\left(f^{n-1}\big(f(x,\omega_1),\omega_2,\omega_3,\ldots\big),x\right)\\ &\le \sum_{k=1}^{n}\rho\left(f^{n-k}\big(f(x,\omega_k),\omega_{k+1},\omega_{k+2},\ldots\big),f^{n-k}(x,\omega_{k+1},\omega_{k+2},\ldots)\right),\end{aligned}$$

where the value $f^n(x,\omega_1,\omega_2,\ldots)$ depends only on x and $(\omega_1,\ldots,\omega_n)$, and by (2) we have

$$\int_{\Omega^\infty}\varrho\left(f^n(x,\omega),f^n(z,\omega)\right)P^\infty(d\omega) \le \lambda^n\varrho(x,z) \quad \text{for } x,z \in X \text{ and } n \in \mathbb{N}.$$

Hence, applying the Fubini theorem, for $x \in X$ and $n \in \mathbb{N}$ we get

$$\begin{aligned}\int_X \rho(x,z)\pi_n^f(x,dz) &= \int_{\Omega^\infty}\rho\left(f^n(x,\omega),x\right)P^\infty(d\omega)\\ &\le \sum_{k=1}^{n}\int_{\Omega^\infty}\rho\Big(f^{n-k}\left(f(x,\omega_1),\omega_2,\omega_3,\ldots\right),f^{n-k}(x,\omega_2,\omega_3,\ldots)\Big)P^\infty\left(d(\omega_1,\omega_2,\ldots)\right)\\ &\le \sum_{k=1}^{n}\lambda^{n-k}\int_\Omega \rho\left(f(x,\omega),x\right)P(d\omega) \le \frac{1}{1-\lambda}\int_\Omega \varrho\left(f(x,\omega),x\right)P(d\omega).\end{aligned}$$

Consequently, for any non-expansive u mapping X into a separable Banach space and for every $x \in X$, $n \in \mathbb{N}$ we obtain

$$\int_X \|u(z)\|\pi_n^f(x,dz) \le \frac{1}{1-\lambda}\int_\Omega \varrho\left(f(x,\omega),x\right)P(d\omega) + \|u(x)\|; \tag{9}$$

moreover,

$$\int_X \|u(z)\|\pi^f(dz) \le \int_X \varrho(x,z)\pi^f(dz) + \|u(x)\|. \tag{10}$$

Let u be a non-expansive mapping of X into a separable Banach space Y. To show that (8) holds for $x \in X$ and $n \in \mathbb{N}$ we may assume that Y is a real space.

Fix $x \in X$, $n \in \mathbb{N}$ and then a $y^* \in Y^*$ such that $\|y^*\| \le 1$ and

$$\left\|\int_X u(z)\pi_n^f(x,dz) - \int_X u\,d\pi^f\right\| = y^*\Big(\int_X u(z)\pi_n^f(x,dz) - \int_X u\,d\pi^f\Big). \tag{11}$$

For every $k \in \mathbb{N}$ the function $\tau_k : \mathbb{R} \to \mathbb{R}$ given by $\tau_k(t) = -k$ for $t \in (-\infty,-k)$, $\tau_k(t) = t$ for $t \in [-k,k]$, $\tau_k(t) = k$ for $t \in (k,\infty)$ is non-expansive and $|\tau_k(t)| \le |t|$ for $t \in \mathbb{R}$. Consequently, since (5) holds for every non-expansive and bounded $u : X \to \mathbb{R}$, for every $k \in \mathbb{N}$ we have

$$\begin{aligned}&\Big|\int_X \tau_k\left(y^*u(z)\right)\pi_n^f(x,dz) - \int_X \tau_k\left(y^*u(z)\right)\pi^f(dz)\Big|\\ &\quad\le \frac{\lambda^n}{1-\lambda}\int_X \varrho\left(f(x,\omega),x\right)P(d\omega)\end{aligned} \tag{12}$$

and, by (9) and (10),

$$\int_X \left|\tau_k\left(y^*u(z)\right)\right| \pi_n^f(x,dz) \le \frac{1}{1-\lambda}\int_\Omega \varrho\left(f(x,\omega),x\right)P(d\omega) + \|u(x)\|,$$

$$\int_X \left|\tau_k\left(y^*u(z)\right)\right| \pi^f(dz) \le \int_X \varrho(x,z)\pi^f(dz) + \|u(x)\|.$$

Hence, taking (3) and (4) into account, applying the Lebesgue dominated convergence theorem and passing with k to the limit in (12) we get

$$\begin{aligned}\Big|\int_X y^*u(z)\pi_n^f(x,dz) &- \int_X y^*u(z)\pi^f(dz)\Big| \\ &\le \frac{\lambda^n}{1-\lambda}\int_X \varrho\left(f(x,\omega),x\right)P(d\omega)\end{aligned}$$

and (8) follows now from (11). □

Proof of the theorem. An easy induction shows that

$$F_n(x) = \int_{\Omega^\infty} F\left(f^n(x,\omega)\right)P^\infty(d\omega),$$

i.e.,

$$F_n(x) = \int_X F(z)\pi_n^f(x,dz) \quad \text{for } x \in X,\ n \in \mathbb{N}.$$

Let L be a Lipschitz constant for F. Putting $u = \frac{1}{L}F$ we have (8), whence

$$\|F_n(x) - \int_X F(z)\pi^f(dz)\| \le \frac{L\lambda^n}{1-\lambda}\int_X \varrho\left(f(x,\omega),x\right)P(d\omega)$$

for $x \in X$ and $n \in \mathbb{N}$. It proves (6) and according to [3, Theorem 2.1(ii)] Eq. (1) has a Lipschitz solution $\varphi : X \to Y$ if and only if (7) holds. □

3. A characterization of the limit distribution

Obviously the problem of characterization of the limit distribution π^f arises. The following theorem provides a characterization via functional equations for some special rv-functions in Hilbert spaces. More exactly, we characterize π^f via a functional equation for its characteristic function $\varphi^f : X \to \mathbb{C}$,

$$\varphi^f(u) = \int_X e^{i(u|z)}\pi^f(dz),$$

cf. [10] by O. K. Zakusilo. Note that any two probability Borel measures on X with the same characteristic function are equal, see [9, Ch. VI, Th. 2.1(2)].

Theorem 3.1. *Assume X is a real separable Hilbert space, $\Lambda : X \to X$ is linear and continuous, $\xi : \Omega \to X$ is $\mathcal{A}$-measurable, and*

$$f(x,\omega) = \Lambda x + \xi(\omega) \quad for\ (x,\omega) \in X \times \Omega.$$

If

$$\|\Lambda\| < 1 \quad and \quad \int_\Omega \|\xi(\omega)\| P(d\omega) < \infty,$$

then the characteristic function of π^f is the only solution $\varphi : X \to \mathbb{C}$ of the equation

$$\varphi(u) = \gamma(u)\varphi(\Lambda^* u) \tag{13}$$

which is continuous at zero and fulfils $\varphi(0) = 1$, where γ stands for the characteristic function of ξ.

Proof. For $n \in \mathbb{N}$ define $\xi_n : \Omega^\infty \to X$ by $\xi_n(\omega_1, \omega_2, \ldots) = \xi(\omega_n)$ and note that $\xi_n,\ n \in \mathbb{N}$, are identically distributed: Denoting by ρ the distribution of ξ we have

$$P^\infty(\xi_n \in B) = P(\xi \in B) = \rho(B)$$

for $n \in \mathbb{N}$ and $B \in \mathcal{B}$. Since

$$f^n(x,\omega) = \Lambda f^{n-1}(x,\omega) + \xi_n(\omega) \quad \text{for } \omega \in \Omega^\infty,$$

and the random variables $\Lambda \circ f^{n-1}(x,\cdot),\ \xi_n$ are independent, we see that

$$\pi_n^f(x,\cdot) = \left(\pi_{n-1}^f(x,\cdot) \circ \Lambda^{-1}\right) * \rho \quad \text{for } n \in \mathbb{N},\ x \in X.$$

Hence, passing to the limit (cf. [9, Ch. III, Th. 1.1]),

$$\pi^f = (\pi^f \circ \Lambda^{-1}) * \rho.$$

Consequently, see also [9, p. 58], for $u \in X$,

$$\begin{aligned}
\varphi^f(u) &= \int_X e^{i(u|z)} \left((\pi^f \circ \Lambda^{-1}) * \rho\right)(dz) \\
&= \int_{X \times X} e^{i(u|x+y)} \left((\pi^f \circ \Lambda^{-1}) \times \rho\right)(d(x,y)) \\
&= \int_X \left(\int_X e^{i(u|x)} \cdot e^{i(u|y)} (\pi^f \circ \Lambda^{-1})(dx)\right) \rho(dy) \\
&= \left(\int_X e^{i(u|x)} (\pi^f \circ \Lambda^{-1})(dx)\right) \left(\int_X e^{i(u|y)} \rho(dy)\right) \\
&= \left(\int_X e^{i(u|\Lambda x)} \pi^f(dx)\right) \gamma(u) = \varphi^f(\Lambda^* u)\gamma(u).
\end{aligned}$$

To prove the uniqueness consider a continuous at zero solution $\varphi : X \to \mathbb{C}$ of (13) such that $\varphi(0) = 1$. Then

$$\varphi(u) = \varphi\left((\Lambda^*)^n u\right) \prod_{k=0}^{n-1} \gamma\left((\Lambda^*)^k u\right) \quad \text{for } n \in \mathbb{N},\ u \in X,$$

and $\lim_{n\to\infty} (\Lambda^*)^n u = 0$ for $u \in X$, whence

$$\varphi(u) = \prod_{n=0}^{\infty} \gamma\left((\Lambda^*)^n u\right) \quad \text{for } u \in X.$$

□

4. Examples

1. Fix an integer $n \geq 2$ and a Lipschitz mapping F of $\mathbb{R}$ into a separable Banach space Y and consider the equation

$$\varphi(x) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi\left(\frac{x+k}{n}\right) + F(x). \tag{14}$$

In this case

$$f(x,\omega) = \frac{1}{n}x + \frac{\omega}{n} \quad \text{for } x \in \mathbb{R},\ \omega \in \{0, 1, \dots n-1\},$$

$\xi(\omega) = \frac{\omega}{n}$ and $P(\{\omega\}) = \frac{1}{n}$ for $\omega \in \{0, 1, \dots n-1\}$. Hence the characteristic function γ of ξ is given by

$$\gamma(u) = \frac{1}{n} \sum_{k=0}^{n-1} \exp\left(iu\frac{k}{n}\right) \quad \text{for } u \in \mathbb{R}.$$

A simple calculation shows that the characteristic function of the uniform distribution $U(0,1)$, i.e. the function $\phi : \mathbb{R} \to \mathbb{C}$ given by

$$\phi(u) = \frac{e^{iu} - 1}{iu} \quad \text{for } u \neq 0, \quad \phi(0) = 1,$$

satisfies

$$\phi(u) = \gamma(u)\phi\left(\frac{1}{n}u\right) \quad \text{for } u \in \mathbb{R}.$$

Hence and from Theorem 3.1 we infer that $\pi^f = U(0,1)$, i.e., $\pi^f(B) = \lambda_1(B \cap [0,1])$ for Borel $B \subset \mathbb{R}$, where λ_1 denotes the one-dimensional Lebesgue measure. Applying now Theorem 2.1 we see that *Eq.* (14) *has a Lipschitz solution* $\varphi : \mathbb{R} \to Y$ *if and only if*

$$\int_{[0,1]} F(z)dz = 0.$$

Cf. [3, Example 2.2].

2. Fix an $\alpha \in (-1,1)$ and a Lipschitz mapping F of $\mathbb{R}$ into a separable Banach space Y and consider the equation

$$\varphi(x) = \int_{\Omega} \varphi\left(\alpha x + \xi(\omega)\right) P(d\omega) + F(x) \tag{15}$$

assuming that $\xi : \Omega \to \mathbb{R}$ is a random variable with the Gaussian law $N(m, \sigma^2)$, i.e. the equation

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(\alpha x + y) e^{-\frac{(y-m)^2}{2\sigma^2}} dy + F(x),$$

where m is a real number and σ is a positive real number. In this case

$$f(x,\omega) = \alpha x + \xi(\omega) \quad \text{for } (x,\omega) \in \mathbb{R} \times \Omega,$$

$$\gamma(u) = \exp\left(imu - \frac{1}{2}\sigma^2 u^2\right) \quad \text{for } u \in \mathbb{R},$$

and, as a simple calculation shows, the function $\phi : \mathbb{R} \to \mathbb{C}$ given by

$$\phi(u) = \exp\left(i\frac{m}{1-\alpha}u - \frac{1}{2}\frac{\sigma^2}{1-\alpha^2}u^2\right)$$

satisfies

$$\phi(u) = \gamma(u)\phi(\alpha u) \quad \text{for } u \in \mathbb{R}.$$

Hence and from Theorem 3.1 we infer that

$$\pi^f = N\left(\frac{m}{1-\alpha}, \frac{\sigma^2}{1-\alpha^2}\right).$$

Applying now Theorem 2.1 we see that *Eq.* (15) *has a Lipschitz solution* $\varphi : \mathbb{R} \to Y$ *if and only if*

$$\int_{\mathbb{R}} F(z) \exp\left(-\left(z - \frac{m}{1-\alpha}\right)^2 / 2\frac{\sigma^2}{1-\alpha^2}\right) dz = 0.$$

In particular (cf., e.g., [5, pp. 299–300]):

2.1. *If* $\alpha \in (-1,1)$ *and* $\xi : \Omega \to \mathbb{R}$ *is a random variable with the Gaussian law* $N(m,\sigma^2)$, *then the equation*

$$\varphi(x) = \int_{\Omega} \varphi\left(\alpha x + \xi(\omega)\right) P(d\omega) + x$$

has a Lipschitz solution $\varphi : \mathbb{R} \to \mathbb{R}$ *if and only if* $m = 0$.

2.2. *If* $\alpha \in (-1,1)$, $\xi : \Omega \to \mathbb{R}$ *is a random variable with the standard Gaussian law* $N(0,1)$, $n \in \mathbb{N}$ and $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$, *then the equation*

$$\varphi(x) = \int_\Omega \varphi\left(\alpha x + \xi(\omega)\right) P(d\omega) + \sum_{k=0}^{n} \alpha_k x^k$$

has a Lipschitz solution $\varphi : \mathbb{R} \to \mathbb{R}$ *if and only if*

$$\sum_{k=0}^{[n/2]} \alpha_{2k} \frac{(2k)!}{k! 2^k (1-\alpha^2)^k} = 0.$$

3. Let X be a real separable Hilbert space. Following [4] denote by $L_1^+(X)$ the set of all linear, symmetric and positive self-mappings of X with finite trace.

Given a linear and symmetric $\Lambda : X \to X$ with $\|\Lambda\| < 1$ and a Lipschitz mapping F of X into a separable Banach space Y consider the equation

$$\varphi(x) = \int_\Omega \varphi\left(\Lambda x + \xi(\omega)\right) P(d\omega) + F(x) \tag{16}$$

assuming now that $\xi : \Omega \to X$ is a random variable with the Gaussian law $N(m, Q)$, where $m \in X$ and $Q \in L_1^+(X)$, and

$$\Lambda Q = Q\Lambda. \tag{17}$$

In this case

$$f(x,\omega) = \Lambda x + \xi(\omega) \quad \text{for } (x,\omega) \in X \times \Omega$$

and (see [4, Sec. 1.2])

$$\gamma(u) = \exp\left(i(m|u) - \frac{1}{2}(Qu|u)\right) \quad \text{for } u \in X.$$

Put

$$A = (I - \Lambda^2)^{-1} Q.$$

Since

$$(I - \Lambda^2)^{-1} = \sum_{n=0}^{\infty} \Lambda^{2n},$$

the operator $(I - \Lambda^2)^{-1}$ is symmetric, and by (17) we have

$$(I - \Lambda^2)^{-1} Q = Q (I - \Lambda^2)^{-1} \quad \text{and} \quad \Lambda A = A\Lambda.$$

Consequently A is symmetric and positive. As Q has finite trace, so has A and $A \in L_1^+(X)$. Moreover, the function $\phi : X \to \mathbb{C}$ given by

$$\phi(u) = \exp\left(i\big((I-\Lambda)^{-1}m|u\big) - \frac{1}{2}(Au|u)\right)$$

satisfies

$$\phi(u) = \gamma(u)\phi(\Lambda u) \quad \text{for } u \in X.$$

Hence and from Theorem 3.1 and [4, Sec. 1.2] we infer that

$$\pi^f = N\big((I-\Lambda)^{-1}m, (I-\Lambda^2)^{-1}Q\big).$$

Applying now Theorem 2.1 we see that *Eq.* (16) *has a Lipschitz solution* $\varphi : X \to Y$ *if and only if*

$$\int_X F dN\big((I-\Lambda)^{-1}m, (I-\Lambda^2)^{-1}Q\big) = 0.$$

In particular, *if* $\Lambda : X \to X$ *is linear and symmetric with* $\|\Lambda\| < 1$ *and* $\xi : \Omega \to X$ *is a random variable with the Gaussian law* $N(m, Q)$, *where* $m \in X$, $Q \in L_1^+(X)$ *and* (17) *holds, then the equation*

$$\varphi(x) = \int_\Omega \varphi\left(\Lambda x + \xi(\omega)\right) P(d\omega) + x$$

has a Lipschitz solution $\varphi : X \to X$ *if and only if* $m = 0$.

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References

[1] Baron, K.: On the convergence in law of iterates of random-valued functions. Aust. J. Math. Anal. Appl. **6**(1), Art. 3, 9 pp. (2009)
[2] Baron, K.: Weak law of large numbers for iterates of random-valued functions. Aequ. Math. **93**, 415–423 (2019)
[3] Baron, K., Morawiec, J.: Lipschitzian solutions to linear iterative equations revisited. Aequ. Math. **91**, 161–167 (2017)

[4] Da Prato, G.: Introduction to stochastic analysis and Malliavin calculus. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie), vol 13. Edizioni della Normale, Pisa (2014)
[5] Dudley, R.M.: Real Analysis and Probability. Cambridge Studies in Advanced Mathematics, vol. 74. Cambridge Univerity Press, Cambridge (2002)
[6] Kapica, R.: Convergence of sequences of iterates of random-valued vector functions. Colloq. Math. **97**, 1–6 (2003)
[7] Kapica, R.: The geometric rate of convergence of random iteration in the Hutchinson distance. Aequ. Math. **93**, 149–160 (2019)
[8] Kuczma, M., Choczewski, B., Ger, R.: Iterative Functional Equations. Encyclopedia of Mathematics and Its Applications, vol. 32. Cambridge University Press, Cambridge (1990)
[9] Parthasarathy, K.R.: Probability Measures on Metric Spaces. Probability and Mathematical Statistics, vol. 3. Academic Press, Inc., New York (1967)
[10] Zakusilo, O.K.: Some properties of random vectors of the form $\sum_{n=0}^{\infty} A^i \xi_i$. Teor. Verojatnost. i Mat. Statist. (Russian) **13**, 59–62 (1975)

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